10-5 Hyperbolas

Main Ideas

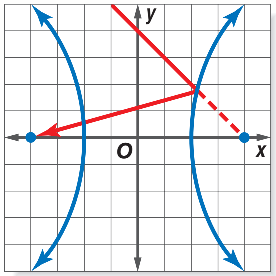
- Write equations of hyperbolas.
- Graph hyperbolas.

New Vocabulary

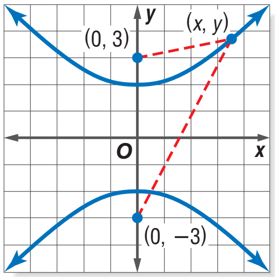
hyperbola
foci
center
vertex
asymptote
transverse axis
conjugate axis

GET READY for the Lesson

A hyperbola is a conic section with the property that rays directed toward one focus are reflected toward the other focus. Notice that, unlike the other conic sections, a hyperbola has two branches.



Equations of Hyperbolas A **hyperbola** is the set of all points in a plane such that the absolute value of the difference of the distances from two fixed points, called the **foci**, is constant.



The hyperbola at the right has foci at (0, 3) and (0, −3). The distances from either of the y-intercepts to the foci are 1 unit and 5 units, so the difference of the distances from any point with coordinates (x, y) on the hyperbola to the foci is 4 or −4 units, depending on the order in which you subtract.

You can use the Distance Formula and the definition of a hyperbola to find an equation of this hyperbola.

The distance between (x, y) and $(0, 3)$ − the distance between (x, y) and $(0, -3)$ $= \pm 4$.

$$\sqrt{x^2 + (y-3)^2} - \sqrt{x^2 + (y+3)^2} = \pm 4$$

$$\sqrt{x^2 + (y-3)^2} = \pm 4 + \sqrt{x^2 + (y+3)^2} \quad \text{Isolate the radicals.}$$

$$x^2 + (y-3)^2 = 16 \pm 8\sqrt{x^2 + (y+3)^2} + x^2 + (y+3)^2$$

$$x^2 + y^2 - 6y + 9 = 16 \pm 8\sqrt{x^2 + (y+3)^2} + x^2 + y^2 + 6y + 9$$

$$-12y - 16 = \pm 8\sqrt{x^2 + (y+3)^2} \quad \text{Simplify.}$$

$$3y + 4 = \pm 2\sqrt{x^2 + (y+3)^2} \quad \text{Divide each side by } -4.$$

$$9y^2 + 24y + 16 = 4[x^2 + (y+3)^2] \quad \text{Square each side.}$$

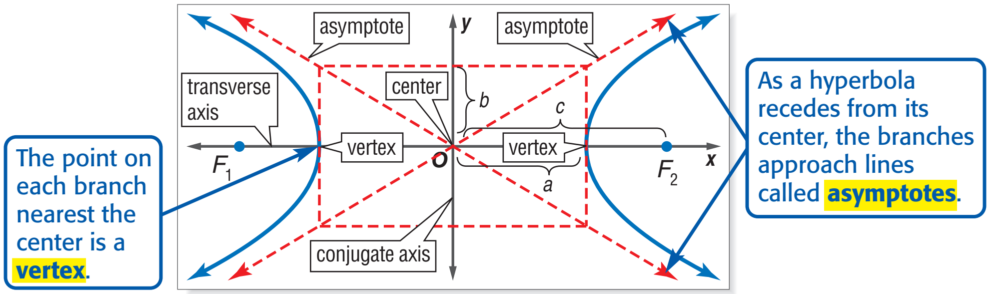
$$9y^2 + 24y + 16 = 4x^2 + 4y^2 + 24y + 36 \quad \text{Distributive Property}$$

$$5y^2 - 4x^2 = 20 \quad \text{Simplify.}$$

$$\frac{y^2}{4} - \frac{x^2}{5} = 1 \quad \text{Divide each side by 20.}$$

An equation of this hyperbola is $\frac{y^2}{4} - \frac{x^2}{5} = 1$.

The diagram below shows the parts of a hyperbola.



The distance from the **center** to a vertex of a hyperbola is a units. The distance from the center to a focus is c units. There are two axes of symmetry. The **transverse axis** is a segment of length $2a$ whose endpoints are the vertices of the hyperbola. The **conjugate axis** is a segment of length $2b$ units that is perpendicular to the transverse axis at the center. The values of a, b, and c are related differently for a hyperbola than for an ellipse. For a hyperbola, $c^2 = a^2 + b^2$.

Reading Math

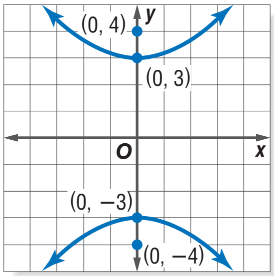
Standard Form In the standard form of a hyperbola, the squared terms are subtracted ($-$). For an ellipse, they are added ($+$).

KEY CONCEPT *Equations of Hyperbolas with Centers at the Origin*

Standard Form of Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Vertices	$(a, 0), (-a, 0)$	$(0, a), (0, -a)$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units
Equations of Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

EXAMPLE Write an Equation for a Graph

1 **Write an equation for the hyperbola shown at the right.**



The center is the midpoint of the segment connecting the vertices, or (0, 0).

The value of a is the distance from the center to a vertex, or 3 units. The value of c is the distance from the center to a focus, or 4 units.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$4^2 = 3^2 + b^2$ $c = 4$, $a = 3$

$16 = 9 + b^2$ Evaluate the squares.

$7 = b^2$ Solve for b^2.

Since the transverse axis is vertical, an equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{7} = 1$.

CHECK Your Progress

1. Write an equation for the hyperbola with vertices at (0, 4) and (0, −4) and foci at (0, 5) and (0, −5).

Real-World EXAMPLE **Write an Equation Given the Foci**

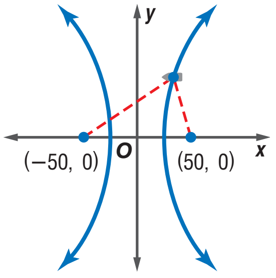
Real-World Link

LORAN stands for *Long Range Navigation.* The LORAN system is generally accurate to within 0.25 nautical mile.

Source: U.S. Coast Guard

2 **NAVIGATION** **The LORAN navigational system is based on hyperbolas. Two stations send out signals at the same time. A ship notes the difference in the times at which it receives the signals. The ship is on a hyperbola with the stations at the foci. Suppose a ship determines that the difference of its distances from two stations is 50 nautical miles. Write an equation for a hyperbola on which the ship lies if the stations are at (−50, 0) and (50, 0).**

First, draw a figure. By studying either of the x-intercepts, you can see that the difference of the distances from any point on the hyperbola to the stations at the foci is the same as the length of the transverse axis, or $2a$. Therefore, $2a = 50$, or $a = 25$. According to the coordinates of the foci, $c = 50$.



Use the values of a and c to determine the value of b for this hyperbola.

$c^2 = a^2 + b^2$	Equation relating a, b, and c for a hyperbola
$50^2 = 25^2 + b^2$	$c = 50$, $a = 25$
$2500 = 625 + b^2$	Evaluate the squares.
$1875 = b^2$	Solve for b^2.

Since the transverse axis is horizontal, the equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Substitute the values for a^2 and b^2. An equation of the hyperbola is $\frac{x^2}{625} - \frac{y^2}{1875} = 1$.

CHECK Your Progress

2. Two microphones are set up underwater 3000 feet apart to observe dolphins. Sound travels under water at 5000 feet per second. One microphone picked up the sound of a dolphin 0.25 second before the other microphone picks up the same sound. Find the equation of the hyperbola that describes the possible locations of the dolphin.

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Graph Hyperbolas It is easier to graph a hyperbola if the asymptotes are drawn first. To graph the asymptotes, use the values of a and b to draw a rectangle with dimensions $2a$ and $2b$. The diagonals of the rectangle should intersect at the center of the hyperbola. The asymptotes will contain the diagonals of the rectangle.

EXAMPLE Graph an Equation in Standard Form

3 **Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Then graph the hyperbola.**

The center of this hyperbola is at the origin. According to the equation, $a^2 = 9$ and $b^2 = 4$, so $a = 3$ and $b = 2$. The coordinates of the vertices are $(3, 0)$ and $(-3, 0)$.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$c^2 = 3^2 + 2^2$ $a = 3, b = 2$

$c^2 = 13$ Simplify.

$c = \sqrt{13}$ Take the square root of each side.

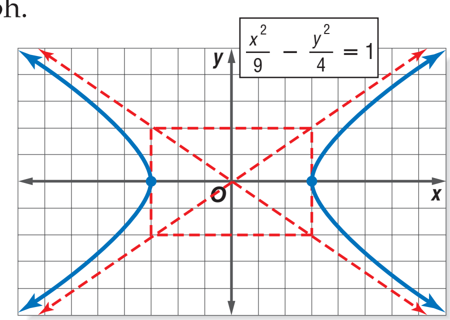
The foci are at $\left(\sqrt{13}, 0\right)$ and $\left(-\sqrt{13}, 0\right)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{2}{3}x$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the hyperbola is centered at the origin, it is symmetric about the y-axis. Therefore, the points at $(-8, 4.9)$, $(-7, 4.2)$, $(-6, 3.5)$, $(-5, 2.7)$, $(-4, 1.8)$, and $(-3, 0)$ lie on the graph.

The hyperbola is also symmetric about the x-axis, so the points at $(-8, -4.9)$, $(-7, -4.2)$, $(-6, -3.5)$, $(-5, -2.7)$, $(-4, -1.8)$, $(4, -1.8)$, $(5, -2.7)$, $(6, -3.5)$, $(7, -4.2)$, and $(8, -4.9)$ also lie on the graph.

Draw a 6-unit by 4-unit rectangle. The asymptotes contain the diagonals of the rectangle. Graph the vertices, which, in this case, are the x-intercepts. Use the asymptotes as a guide to draw the hyperbola that passes through the vertices and the other points. The graph does not intersect the asymptotes.



Study Tip

Graphing Calculator

You can graph a hyperbola on a graphing calculator. Similar to an ellipse, first solve the equation for y. Then graph the two equations that result on the same screen.

CHECK Your Progress

3. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $\frac{x^2}{49} - \frac{y^2}{25} = 1$. Then graph the hyperbola.

So far, you have studied hyperbolas that are centered at the origin. A hyperbola may be translated so that its center is at (h, k). This corresponds to replacing x by $x - h$ and y by $y - k$ in both the equation of the hyperbola and the equations of the asymptotes.

KEY CONCEPT *Equations of Hyperbolas with Centers at (h, k)*

Standard Form of Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Equations of Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

When graphing a hyperbola given an equation that is not in standard form, begin by rewriting the equation in standard form.

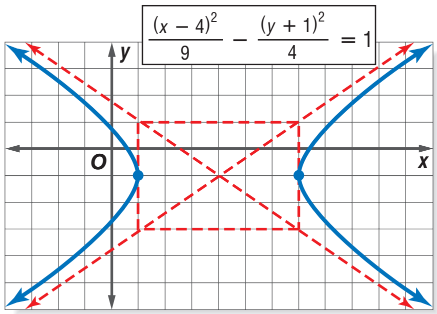
EXAMPLE Graph an Equation Not in Standard Form

4 **Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $4x^2 - 9y^2 - 32x - 18y + 19 = 0$. Then graph the hyperbola.**

Complete the square for each variable to write this equation in standard form.

$4x^2 - 9y^2 - 32x - 18y + 19 = 0$	Original equation
$4(x^2 - 8x + \blacksquare) - 9(y^2 + 2y + \blacksquare) = -19 + 4(\blacksquare) - 9(\blacksquare)$	Complete the squares.
$4(x^2 - 8x + 16) - 9(y^2 + 2y + 1) = -19 + 4(16) - 9(1)$	
$4(x - 4)^2 - 9(y + 1)^2 = 36$	Write the trinomials as perfect squares.
$\dfrac{(x-4)^2}{9} - \dfrac{(y+1)^2}{4} = 1$	Divide each side by 36.

The graph of this hyperbola is the graph from Example 3 translated 4 units to the right and down 1 unit. The vertices are at $(7, -1)$ and $(1, -1)$, and the foci are at $(4 + \sqrt{13}, -1)$ and $(4 - \sqrt{13}, -1)$. The equations of the asymptotes are $y + 1 = \pm\frac{2}{3}(x - 4)$.



CHECK Your Progress

4. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $9x^2 - 25y^2 - 36x - 50y - 214 = 0$. Then graph the hyperbola.

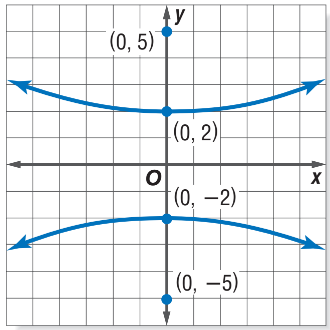
CHECK Your Understanding

Example 1 (pp. 591–592)

1. Write an equation for the hyperbola shown at right.

Example 2 (p. 592)

2. A hyperbola has foci at $(4, 0)$ and $(-4, 0)$. The value of a is 1. Write an equation for the hyperbola.



3. ASTRONOMY Comets or other objects that pass by Earth or the Sun only once and never return may follow hyperbolic paths. Suppose a comet's path can be modeled by a branch of the hyperbola with equation $\frac{y^2}{225} - \frac{x^2}{400} = 1$. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola. Then graph the hyperbola.

Examples 3, 4
(pp. 593, 594)

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

4. $\frac{y^2}{18} - \frac{x^2}{20} = 1$

5. $\frac{(y+6)^2}{20} - \frac{(x-1)^2}{25} = 1$

6. $x^2 - 36y^2 = 36$

7. $5x^2 - 4y^2 - 40x - 16y - 36 = 0$

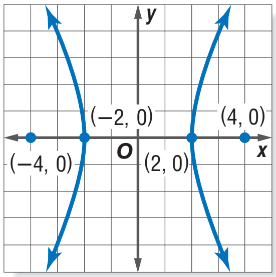
Exercises

HOMEWORK HELP

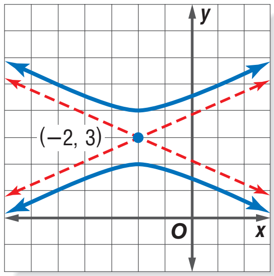
For Exercises	See Examples
8–11	1
12–15	2
16–21	3
22–25	4

Write an equation for each hyperbola.

8.



9.



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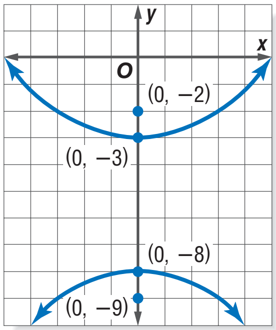
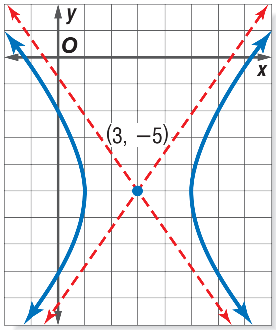


11.



Write an equation for the hyperbola that satisfies each set of conditions.

12. vertices $(-5, 0)$ and $(5, 0)$, conjugate axis of length 12 units

13. vertices $(0, -4)$ and $(0, 4)$, conjugate axis of length 14 units

14. vertices $(9, -3)$ and $(-5, -3)$, foci $\left(2 \pm \sqrt{53}, -3\right)$

15. vertices $(-4, 1)$ and $(-4, 9)$, foci $\left(-4, 5 \pm \sqrt{97}\right)$

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

16. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

17. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

18. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

19. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

20. $\frac{(y-4)^2}{16} - \frac{(x+2)^2}{9} = 1$

21. $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$

22. $x^2 - 2y^2 = 2$

23. $x^2 - y^2 = 4$

24. $y^2 = 36 + 4x^2$

25. $6y^2 = 2x^2 + 12$

26. $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

27. $\frac{(x+6)^2}{36} - \frac{(y+3)^2}{9} = 1$

28. $y^2 - 3x^2 + 6y + 6x - 18 = 0$

29. $4x^2 - 25y^2 - 8x - 96 = 0$

30. Find an equation for a hyperbola centered at the origin with a horizontal transverse axis of length 8 units and a conjugate axis of length 6 units.

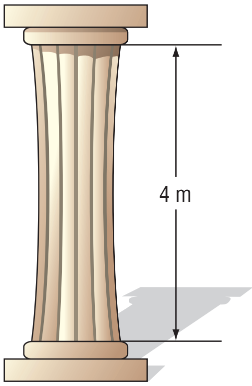
31. What is an equation for the hyperbola centered at the origin with a vertical transverse axis of length 12 units and a conjugate axis of length 4 units?

Real-World Link

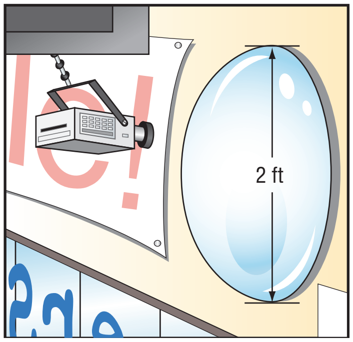
The Parthenon, originally constructed in 447 B.C., has 46 columns around its exterior perimeter.

Source: www.mlahanas.de/Greeks/Arts/Parthenon.htm

32. **STRUCTURAL DESIGN** An architect's design for a building includes some large pillars with cross sections in the shape of hyperbolas. The curves can be modeled by the equation $\frac{x^2}{0.25} - \frac{y^2}{9} = 1$, where the units are in meters. If the pillars are 4 meters tall, find the width of the top of each pillar and the width of each pillar at the narrowest point in the middle. Round to the nearest centimeter.

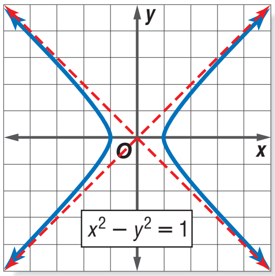


33. **PHOTOGRAPHY** A curved mirror is placed in a store for a wide-angle view of the room. The equation $\frac{x^2}{1} - \frac{y^2}{3} = 1$ models the curvature of the mirror. A small security camera is placed 3 feet from the vertex of the mirror so that a diameter of 2 feet of the mirror is visible. If the back of the room lies on $x = -18$, what width of the back of the room is visible to the camera?



NONRECTANGULAR HYPERBOLA **For Exercises 34–37, use the following information.**

A hyperbola with asymptotes that are not perpendicular is called a **nonrectangular hyperbola**. Most of the hyperbolas you have studied so far are nonrectangular. A **rectangular hyperbola** is a hyperbola with perpendicular asymptotes. For example, the graph of $x^2 - y^2 = 1$ is a rectangular hyperbola. The graphs of equations of the form $xy = c$, where c is a constant, are rectangular hyperbolas with the coordinate axes as their asymptotes.



34. Plot some points and use them to graph the equation. Be sure to consider negative values for the variables.
35. Find the coordinates of the vertices of the graph of $xy = 2$.
36. Graph $xy = -2$.
37. Describe the transformations that can be applied to the graph of $xy = 2$ to obtain the graph of $xy = -2$.

EXTRA PRACTICE

See pages 913, 935.

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H.O.T. Problems

38. **OPEN ENDED** Find and graph a counterexample to the following statement.
If the equation of a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *then* $a^2 \geq b^2$.

39. **REASONING** Describe how the graph of $y^2 - \frac{x^2}{k^2} = 1$ changes as $|k|$ increases.

40. **CHALLENGE** A hyperbola with a horizontal transverse axis contains the point at (4, 3). The equations of the asymptotes are $y - x = 1$ and $y + x = 5$. Write the equation for the hyperbola.

41. ***Writing in Math*** Explain how hyperbolas and parabolas are different. Include differences in the graphs of hyperbolas and parabolas and differences in the reflective properties of hyperbolas and parabolas.

STANDARDIZED TEST PRACTICE

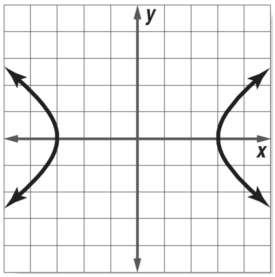
42. ACT/SAT The foci of the graph are at $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$. Which equation does the graph represent?

A $\frac{x^2}{9} - \frac{y^2}{4} = 1$

B $\frac{x^2}{3} - \frac{y^2}{2} = 1$

C $\frac{x^2}{3} - \frac{y^2}{\sqrt{13}} = 1$

D $\frac{x^2}{9} - \frac{y^2}{13} = 1$



43. REVIEW To begin a game, Nate must randomly draw a red, blue, green, or yellow game piece, and a tile from a group of 26 tiles labeled with all the letters of the alphabet. What is the probability that Nate will draw the green game piece and a tile with a letter from his name?

F $\frac{1}{26}$

G $\frac{1}{13}$

H $\frac{3}{52}$

J $\frac{1}{2}$

Spiral Review

Write an equation for the ellipse that satisfies each set of conditions. (Lesson 10-4)

44. endpoints of major axis at (1, 2) and (9, 2), endpoints of minor axis at (5, 1) and (5, 3)

45. major axis 8 units long and parallel to y-axis, minor axis 6 units long, center at $(-3, 1)$

46. foci at (5, 4) and $(-3, 4)$, major axis 10 units long

47. Find the center and radius of the circle with equation $x^2 + y^2 - 10x + 2y + 22 = 0$. Then graph the circle. (Lesson 10-3)

Solve each equation by factoring. (Lesson 5-2)

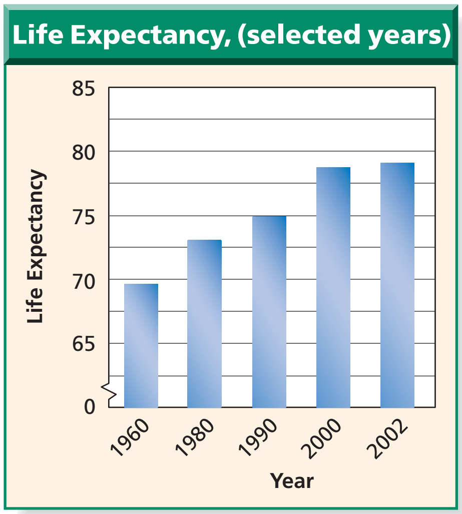
48. $x^2 + 6x + 8 = 0$

49. $2q^2 + 11q = 21$

50. LIFE EXPECTANCY Refer to the graph at the right. What was the average rate of change of life expectancy from 1960 to 2002? (Lesson 2-3)

51. Solve $|2x + 1| = 9$. (Lesson 1-4)

52. Simplify $7x + 8y + 9y - 5x$. (Lesson 1-2)



Source: National Center for Health Statistics

GET READY for the Next Lesson

PREREQUISITE SKILL **Each equation is of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Identify the values of A, B, and C.** (Lesson 6-1)

53. $2x^2 + 3xy - 5y^2 = 0$

54. $-3x^2 + xy + 2y^2 + 4x - 7y = 0$

55. $x^2 - 4x + 4y + 2 = 0$

56. $-xy - 2x - 3y + 6 = 0$